ON THE EXPONENT OF TENSOR CATEGORIES COMING FROM FINITE GROUPS

BY

SONIA NATALE*

*Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba
CIEM – CONICET, (5000) Ciudad Universitaria, Córdoba, Argentina
e-mail: natale@mate.uncor.edu
URL: http://www.mate.uncor.edu/natale*

ABSTRACT

We describe the exponent of a group-theoretical fusion category $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$ associated to a finite group G in terms of group cohomology. We show that the exponent of $\mathcal{C}$ divides both $e(\omega)\exp G$ and $(\exp G)^2$, where $e(\omega)$ is the cohomological order of the 3-cocycle ω. In particular, $\exp\mathcal{C}$ divides $(\dim\mathcal{C})^2$.

1. Introduction and main results

Throughout this note we shall work over an algebraically closed base field k of characteristic zero. The notion of (quasi)exponent of a finite-dimensional Hopf algebra H has been introduced in a series of papers by Etingof and Gelaki [11, 12] extending previous work of Kashina [18, 19]. By definition, the exponent of H is the least integer N for which

$$m_N(\mathrm{id} \otimes \mathcal{S}^{-2} \otimes \cdots \otimes \mathcal{S}^{-2N+2})\Delta_N = \epsilon 1,$$

where $\Delta_N\colon H \to H^{\otimes N}$ and $m_N\colon H^{\otimes N} \to H$ are the iterated comultiplication and multiplication maps, respectively. This gives a non-commutative analogue of the exponent of a group.

It was conjectured, in the context of semisimple Hopf algebras, that the order of a certain power map divides the dimension of H. In terms of the exponent, the conjecture can be stated as follows:

CONJECTURE 1.1 ([18]): *If H is a semisimple Hopf algebra over k, then the exponent of H divides the dimension of H.*

This problem has an affirmative answer in a number of cases, but the general answer is still not known. Etingof and Gelaki have proved several basic and important properties and characterizations of the exponent, in particular, they have shown that the exponent divides $(\dim H)^3$. One important property of the exponent is its gauge invariance: that is, the exponent does not depend on the Hopf algebra itself but only on its tensor category of representations. Generalizing the definitions for finite dimensional Hopf algebras, Etingof introduced the quasi-exponent of a finite rigid tensor category $\mathcal{C}$ in [10].

The main goal of this paper is to describe the exponent of a large class of semisimple Hopf algebras, which exhausts all known examples, in terms of group cohomology. Actually, this class consists not only of semisimple Hopf algebras but also of semisimple quasi-Hopf algebras. Our results will imply that the exponent of H divides $(\dim H)^2$ for all H in this class.

Group-theoretical fusion categories were introduced by Ostrik in [28]. Let G be a finite group, and let $\omega\colon G \times G \times G \to k^\times$ be a normalized 3-cocycle. Let also $F \subseteq G$ be a subgroup and $\tau\colon F \times F \to k^\times$ a normalized 2-cochain $\omega|_{F \times F \times F} = d\tau$. A **group-theoretical category** is a tensor category equivalent to the category $\mathcal{C}(G, \omega, F, \alpha)$ of $k_\alpha F$-bimodules in the category Vec_ω^G of G-graded vector spaces with associativity given by ω. A (quasi)-Hopf algebra H is called group theoretical if $\mathrm{Rep}\, H$ is.

Recall that the **global dimension** of $\mathcal{C}$, denoted $\dim \mathcal{C}$, is the sum of squares of the categorical dimensions of simple objects in $\mathcal{C}$. If $\mathrm{Rep}\, H \simeq \mathcal{C}(G, \omega, F, \alpha)$, then $\dim \mathcal{C} = \dim H = |G|$.

It is an open question whether every semisimple Hopf algebra over k is group-theoretical or not [14]. Every group-theoretical category is equivalent to the category of representations of a quasi-Hopf algebra. The explicit structure, up to gauge equivalence, of group theoretical quasi-Hopf algebras was given in [26], where other invariants, the Frobenius–Schur indicators, were computed in terms of the group-theoretical data G, ω, F, τ.

Using a result of Schauenburg on the center of a bimodule category, it was shown in [25] that a quasi-Hopf algebra H is group theoretical if and only if its quantum double is gauge equivalent to a Dijkgraaf–Pasquier–Roche quasi-Hopf algebra $D^\omega G$ [7].

In this paper we prove the following characterization of the exponent of a group-theoretical category.

THEOREM 1.2: *The exponent of $\mathcal{C} = \mathcal{C}(G, \omega, F, \alpha)$ divides the modified exponent*

$$\exp_\omega G := \text{l.c.m.}[e(\omega_g)|g| : g \in G],$$

and, moreover, $\exp \mathcal{C} = \exp_\omega G$ in either of the following cases:
  (i) *$|G|$ is odd,*
  (ii) *$\mathcal{C}$ admits a fiber functor.*

Here ω_g is the restriction of ω to the subgroup generated by g and $e(\omega_g)$ is its cohomological order. Condition (ii) means that $\mathcal{C}$ is the category of representations a Hopf algebra. Theorem 1.2 allows us to give necessary and sufficient conditions for $\exp \mathcal{C}$ to divide $\dim \mathcal{C}$ when the last is odd or $\mathcal{C}$ admits a fiber functor. See Theorems 5.12, 5.18.

The following theorems are proved as a consequence of this characterization.

THEOREM 1.3: *The exponent of the twisted quantum double $D^\omega G$ divides $(\exp G)^2$.*

In particular, the exponent conjecture holds true for all semisimple quasi-Hopf algebras which are gauge equivalent to a twisted Drinfeld double $D^\omega G$. We also prove that for the quasi-Hopf algebra $D^\omega G$, the order of the element β divides the exponent of G.

THEOREM 1.4: *Let $\mathcal{C} \simeq \mathcal{C}(G, \omega, F, \tau)$ be a group-theoretical fusion category. Then*
  (i) *$\exp G$ divides $\exp \mathcal{C}$.*
  (ii) *$\exp \mathcal{C}$ divides $e(\omega)\exp G$. In particular, $\exp \mathcal{C}$ divides $(\dim \mathcal{C})^2$.*
  (iii) *$\exp \mathcal{C}$ divides $(\exp G)^2$.*
  (iv) *$\exp \mathcal{C}$ and $\dim \mathcal{C}$ have the same prime divisors.*

The proof relies on the characterization result in [25]. We note that part (iv) has been recently established in [20] for any semisimple Hopf algebra H.

In particular, it follows from Theorem 1.3 that the exponent of A divides $(\dim A)^2$ for all bicrossed products arising from exact factorizations of finite groups [25, Theorem 1.3] and all their twistings, that is, for all semisimple Hopf algebras which are twist equivalent to some A that fits into an abelian exact sequence

(1.1) $$k \to k^\Gamma \to A \to kF \to k,$$

where F and Γ form a **matched pair** of finite groups; see [23, 24].

In this case, we show that $\exp A$ divides $\exp \text{Opext}(F, \Gamma) \exp G$, where $G = F \bowtie \Gamma$ is the factorizable group determining the matched pair and $\text{Opext}(F, \Gamma)$ is the abelian group classifying all extensions (1.1). See Corollary 5.24. Among semisimple Hopf algebras arising from abelian extensions, the conjecture on the exponent had been established under additional restrictions [19].

In the context of abelian extensions like (1.1) we also obtain, as an application of results of Masuoka [24], a result that is of independent interest: we prove a Hopf algebra generalization of the Schur–Zassenhauss Theorem for finite groups. Namely, suppose that the orders of Γ and F are relatively prime. Then, after twisting the multiplication and comultiplication if necessary, A is equivalent to the split extension $k^\Gamma \# kF$. See Proposition 5.22.

The paper is organized as follows: in Sections 2 and 3 we recall some properties of the exponent of a fusion category and of quasi-Hopf algebras, respectively. In Section 4 we review results of Altschuler and Coste on Drinfeld and ribbon elements and prove, under certain assumptions, some results on the powers of the Drinfeld element that generalize those in [11]. Finally, in Section 5, we prove our main results, using the ribbon element for a twisted Drinfeld double; Subsections 5.1 and 5.2 concern particularly the Hopf algebra case.

After submission of this paper, the author knew about the preprint [27], where a related invariant, the Frobenius-Schur exponent, is introduced and studied.

2. Exponent of a tensor category

In this section we recall the notion of (quasi-)exponent of a finite tensor category introduced by Etingof in [10, Section 6]. This generalization is based on the results on the (quasi-)exponent of a finite dimensional Hopf algebra found in the papers [11, 12].

Let $\mathcal{C}$ be a finite rigid tensor category over k and let $\mathcal{Z}(\mathcal{C})$ be its Drinfeld's center, which is a braided tensor category with respect to a canonical braiding $\beta_{UV} \colon U \otimes V \to V \otimes U$. In the paper [10], the **quasi-exponent** $\operatorname{qexp} \mathcal{C}$ of $\mathcal{C}$ is defined as the smallest integer N such that $(\beta^2)^N$ is unipotent in $\mathcal{Z}(\mathcal{C})$. Here, β^2 is the natural automorphism $\beta_{VU} \beta_{UV} \colon U \otimes V \to U \otimes V$.

The notion of quasi-exponent of a finite tensor category gives, by restriction, a notion of quasi-exponent of a finite dimensional quasi-Hopf algebra H which, by definition, is a gauge invariant of H.

If $\mathcal{C}$ is a **fusion** category, then the quasi-exponent of $\mathcal{C}$ is called the **exponent** of $\mathcal{C}$ and denoted $\exp\mathcal{C}$. We shall be interested in fusion categories of the form $\mathcal{C} = \operatorname{Rep} H$, where H is a finite dimensional semisimple quasi-Hopf algebra. In this case $\exp\mathcal{C}$ will be called the **exponent of** H and denoted $\exp H$.

It follows from [10, Proposition 6.3] that the exponent of $\mathcal{C} = \operatorname{Rep} H$ satisfies the following:

$$(2.1) \qquad\qquad \exp\mathcal{C} = \exp\mathcal{Z}(\mathcal{C});$$

$$(2.2) \qquad\qquad \exp\mathcal{C} \text{ equals the order of } \beta^2.$$

Moreover, [10, Theorem 5.1] implies that $\exp\mathcal{C}$ is finite.

3. Semisimple quasi-Hopf algebras

Let $(H, \Delta, \epsilon, \phi, \mathcal{S}, \alpha, \beta)$ be a finite dimensional semisimple quasi-Hopf algebra [8] (later on indicated by H for short). Here, $\phi \in (H^{\otimes 3})^{\times}$ is the **associator**, $\mathcal{S} \colon H \to H^{\mathrm{op}}$ is the **quasi-antipode** and $\alpha, \beta \in H$ are related to $\mathcal{S}$ by

$$(3.1) \qquad \mathcal{S}(h_1)\alpha h_2 = \epsilon(h)\alpha, \quad h_1\beta\mathcal{S}(h_2) = \epsilon(h)\beta, \quad \forall h \in H;$$

$$(3.2) \qquad \phi^{(1)}\beta\mathcal{S}(\phi^{(2)})\alpha\phi^{(3)} = 1 = \mathcal{S}(\phi^{(-1)})\alpha\phi^{(-2)}\beta\mathcal{S}(\phi^{(-3)}),$$

where $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$ and $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)}$.

The category $\operatorname{Rep} H =: \operatorname{Rep}(H, \phi)$ is a fusion category of global dimension $\dim\mathcal{C} = \dim H$ with associativity given by the action of ϕ.

Let H_1 and H_2 be finite dimensional semisimple quasi-Hopf algebras. The categories $\operatorname{Rep} H_1$ and $\operatorname{Rep} H_2$ are tensor equivalent if and only if H_1 and H_2 are gauge equivalent [13]; that is, if and only if there exists an invertible normalized element $F \in H_1 \otimes H_1$ (a **gauge transformation**) such that $(H_1)_F$ and H_2 are isomorphic as quasi-bialgebras, where $(H_1)_F$ is the quasi-Hopf algebra $(H_1, \Delta_F, \epsilon, \phi_F, \mathcal{S}_F, \alpha_F, \beta_F)$, such that

$$\Delta_F(h) = F\Delta(h)F^{-1}, \quad h \in H,$$
$$\phi_F = (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\phi(\Delta \otimes \mathrm{id})(F^{-1})(F^{-1} \otimes 1),$$
$$\alpha_F = \mathcal{S}(F^{(-1)})\alpha F^{(-2)}, \quad \beta_F = F^{(1)}\beta\mathcal{S}(F^{(2)});$$

with the notation $F = F^{(1)} \otimes F^{(2)}$, $F^{-1} = F^{(-1)} \otimes F^{(-2)}$.

There is also a notion of **quasitriangular** quasi-Hopf algebra, requiring the existence of an invertible R-matrix $R \in H \otimes H$. When H is quasitriangular the category $\operatorname{Rep} H$ is a braided tensor category with braiding given by the action of R.

The center $\mathcal{Z}(\mathrm{Rep}\,H)$ is equivalent to the representation category of the quantum double $D(H)$ [21, 15]: this is a quasitriangular semisimple quasi-Hopf algebra with underlying vector space $H^* \otimes H$ and canonical R-matrix

$$(3.3) \qquad\qquad \mathcal{R} = \sum_i h_i \otimes D(h^i),$$

where $(h_i)_i$ is a basis of H and $(h^i)_i$ is the dual basis.

Since the element $\mathcal{R}_{21}\mathcal{R} \in D(H) \otimes D(H)$ implements the natural isomorphism β^2 in the category $\mathrm{Rep}\,D(H) \simeq \mathcal{Z}(\mathrm{Rep}\,H)$, the results in [10] imply the following lemma.

LEMMA 3.1: *The order of $\mathcal{R}_{21}\mathcal{R}$ is finite and equals the exponent of $\mathrm{Rep}\,H$.*

3.1. TWISTED QUANTUM DOUBLES.   Let G be a finite group and ω a normalized 3-cocycle on G. The identity element in G will be denoted by e. Let H be the quasi-Hopf algebra (k^G, ω) of k-valued functions on G with associator $\omega \in k^G \otimes k^G \otimes k^G$. Then the quantum double of H is a quasitriangular quasi-Hopf algebra isomorphic to the Dijkgraaf–Pasquier–Roche quasi-Hopf algebra $D^\omega G$ [21, 7].

This quasi-Hopf algebra is defined on the vector space $k^G \otimes kG$ as follows. Consider the maps $\theta, \gamma \colon G \times G \to (k^G)^\times$,

$$\theta(x,y) = \sum_{g \in G} \theta_g(x,y)e_g, \quad \gamma(x,y) = \sum_{g \in G} \gamma_g(x,y)e_g,$$

where $e_g \in k^G$ are the canonical idempotents: $e_g(h) = \delta_{g,h}$, $g, h \in G$, and

$$(3.4) \qquad \theta_g(x,y) = \frac{\omega(g,x,y)\omega(x,y,(xy)^{-1}g(xy))}{\omega(x,x^{-1}gx,y)},$$

$$(3.5) \qquad \gamma_g(x,y) = \frac{\omega(x,y,g)\omega(g,g^{-1}xg,g^{-1}yg)}{\omega(x,g,g^{-1}yg)}.$$

Then $D^\omega G$ is as an algebra the crossed product $k^G \#_\theta kG$, with respect to the adjoint action, and it is the crossed product $k^{G\gamma} \# kG$ as a coalgebra, with respect to the trivial coaction.

A basis of $D^\omega G$ consists of the elements $e_g \# x$, $g, x \in G$. The multiplication and comultiplication are explicitly determined by

$$(3.6) \qquad (e_g \# x)(e_h \# y) = \delta_{g,xhx^{-1}}\theta_g(x,y)e_g \# xy,$$

$$(3.7) \qquad \Delta(e_g \# x) = \sum_{st=g} \gamma_x(s,t)e_s \# x \otimes e_t \# x.$$

The unit element is $1 := 1\#e = \sum_{g \in G} e_g \#e$ and the counit and antipode are determined by

(3.8) $\epsilon(e_g \# x) = \delta_{g,e},$

(3.9) $\mathcal{S}(e_g \# x) = \theta_{g^{-1}}(x, x^{-1})^{-1} \gamma_x(g, g^{-1})^{-1} e_{x^{-1}g^{-1}x} \# x^{-1},$

with $\alpha = 1$ and $\beta = \sum_g \omega(g, g^{-1}, g) e_g$.

This is a quasitriangular quasi-Hopf algebra with associator and R-matrix given, respectively, by

(3.10) $\phi = \sum_{a,b,c} \omega(a, b, c)^{-1} e_a \otimes e_b \otimes e_c, \quad \mathcal{R} = \sum_g e_g \otimes g.$

It is also known that in the twisted Drinfeld double, β is an invertible element with inverse $\mathcal{S}(\beta) = \beta^{-1}$. Moreover, β implements $\mathcal{S}^2$ by conjugation; that is, for all elements $h \in D^\omega G$ we have the relation

(3.11) $\mathcal{S}^2(h) = \beta^{-1} h \beta.$

Recall that an element $a \in D^\omega G$ is called **group-like** if $\Delta(a) = a \otimes a$. Axiom (3.1) for the antipode, combined with the fact that $\alpha = 1$ in $D^\omega G$, implies that the set of non-zero group-like elements form a subgroup of the group of units of $D^\omega G$, denoted $G(D^\omega G)$, and we have $\mathcal{S}(a) = a^{-1}$, for all $a \in G(D^\omega G)$.

The group-like elements in twisted Drinfeld doubles have been completely described in [22, Proposition 3.2]: an element $a \in D^\omega G$ is group-like if and only if there exist elements $x \in G$ and $f \in k^G$ such that

$$\gamma_x = df, \quad \text{and} \quad a = f \# x.$$

Here $df \colon G \to k$ denotes the coboundary of f given by

$$df(g, h) = f(g) f(h) f(gh)^{-1}.$$

4. Drinfeld elements and ribbon elements

Let (H, ϕ, R) be a quasitriangular quasi-Hopf algebra. Let $u \in H$ be the element defined by

$$u := \mathcal{S}(\phi^{(-2)} \beta \mathcal{S}(\phi^{(-3)})) \mathcal{S}(R^j) \alpha R_j \phi^{(-1)} = m_{21}(\mathrm{id} \otimes \mathcal{S})((\alpha \otimes 1) Rp),$$

where $p = p_R = \phi^{(-1)} \otimes \phi^{(-2)} \beta \mathcal{S}(\phi^{(-3)}) \in H \otimes H$ is the special element related to a canonical adjunction formula in Rep H [8, 15].

The element u has been introduced by Altschuler and Coste in [1] generalizing the Drinfeld element for quasitriangular Hopf algebras [9]. It satisfies

$$\mathcal{S}^2(h) = uhu^{-1},$$

for all $h \in H$ [1, Section 3]. Here, and in what follows, we are using the notation $R = R_j \otimes R^j$, assuming a summation symbol over repeated indexes. The action of u on finite dimensional representations of H gives a canonical isomorphism between the dual and double dual functors.

Remark 4.1: The Drinfeld element u satisfies the equation [1, (3.9)]

$$\mathcal{S}(\alpha)u = \mathcal{S}(R^j)\alpha R_j.$$

In particular, when $\alpha = 1$, we get $u = \mathcal{S}(R^j)R_j$, which coincides with the formula given by Drinfeld in the Hopf algebra case.

Let us denote $\widetilde{R} = (\alpha \otimes 1)Rp \in H \otimes H$. Let $n \geq 1$ be an integer. Following [11, 12] we define an element $\widetilde{R}_n \in H \otimes H$ by the formula

$$\begin{aligned}
\widetilde{R}_n &= \widetilde{R}(\mathrm{id} \otimes \mathcal{S}^2)(\widetilde{R}) \cdots (\mathrm{id} \otimes \mathcal{S}^{2n-2})(\widetilde{R}) \\
&= \widetilde{R}_{i_1} \widetilde{R}_{i_2} \cdots \widetilde{R}_{i_n} \otimes \widetilde{R}^{i_1} \mathcal{S}^2(\widetilde{R}^{i_2}) \cdots \mathcal{S}^{2n-2}(\widetilde{R}^{i_n}),
\end{aligned}$$

where $\widetilde{R} = \widetilde{R}_{i_1} \otimes \widetilde{R}^{i_1} = \widetilde{R}_{i_2} \otimes \widetilde{R}^{i_2} = \cdots = \widetilde{R}_{i_n} \otimes \widetilde{R}^{i_n}$.

As in the Hopf algebra case, this element is related to the powers of the Drinfeld element. The following lemma gives the precise relation.

LEMMA 4.2: *We have* $u^n = m_{21}(\mathrm{id} \otimes \mathcal{S})(\widetilde{R}_n)$.

Proof: The proof is by induction on n. If $n = 1$, there is nothing to prove. Let $n \geq 2$. We have

$$\begin{aligned}
m_{21}(\mathrm{id} \otimes \mathcal{S})(\widetilde{R}_{n+1}) &= \mathcal{S}^{2n+1}(\widetilde{R}^{i_{n+1}}) \cdots \mathcal{S}(\widetilde{R}^{i_1})\widetilde{R}_{i_1} \cdots \widetilde{R}_{i_{n+1}} \\
&= \mathcal{S}^{2n+1}(\widetilde{R}^{i_{n+1}}) \cdots \mathcal{S}^3(\widetilde{R}^{i_2})u\widetilde{R}_{i_2} \cdots \widetilde{R}_{i_{n+1}} \\
&= \mathcal{S}^{2n+1}(\widetilde{R}^{i_{n+1}}) \cdots \mathcal{S}^3(\widetilde{R}^{i_2})\mathcal{S}^2(\widetilde{R}_{i_2}) \cdots \mathcal{S}^2(\widetilde{R}_{i_{n+1}})u \\
&= \mathcal{S}^2(\mathcal{S}^{2n-1}(\widetilde{R}^{i_{n+1}}) \cdots \mathcal{S}(\widetilde{R}^{i_2})\widetilde{R}_{i_2} \cdots \widetilde{R}_{i_{n+1}})u \\
&= \mathcal{S}^2(u^n)u = u^n u = u^{n+1},
\end{aligned}$$

by induction, and using that $\mathcal{S}^2(u) = u$. This proves the lemma. ∎

Remark 4.3: Suppose that $\alpha = 1$. By Remark 4.1, $u = \mathcal{S}(R^j)R_j$. Then it is not difficult to show by induction on n that in this case we have $u^n = m_{21}(\mathrm{id} \otimes \mathcal{S})(R_n)$, where R_n is defined by

$$R_n = R(\mathrm{id} \otimes \mathcal{S}^2)(R) \cdots (\mathrm{id} \otimes \mathcal{S}^{2n-2})(R),$$

as in the Hopf algebra case.

Unlike in the semisimple Hopf algebra case, it may happen that H is semisimple but u is not a ribbon element in H. Suppose α is invertible in H. According to the definition in 4.1 and Remark on page 13 in [1], a ribbon element in H is the same as a central element $v \in H$ satisfying

(4.1) $\qquad\qquad v^2 = u\mathcal{S}(u);$

(4.2) $\qquad\qquad \mathcal{S}(v) = v;$

(4.3) $\qquad\qquad \epsilon(v) = 1;$

(4.4) $\qquad\qquad \Delta(v) = (v \otimes v)(R_{21}R)^{-1} = (R_{21}R)^{-1}(v \otimes v).$

For a finite group G with normalized 3-cocycle ω, the expression for the Drinfeld element in the twisted quantum double $D^\omega G$ is the following:

(4.5) $\qquad\qquad u = \sum_{g \in G} \omega(g, g^{-1}, g)^{-2} e_g \# g^{-1}.$

Moreover, $D^\omega G$ is a ribbon quasi-Hopf algebra with ribbon element v given by

(4.6) $\qquad\qquad v = \sum_{g \in G} \omega(g, g^{-1}, g)^{-1} e_g \# g^{-1},$

and the following relation holds:

(4.7) $\qquad\qquad\qquad\qquad v = \beta u.$

See [1, Section 5].

The action of the ribbon element v on irreducible representations gives the (twisted) **modular invariant** matrix T studied in various papers, see for instance [4, 6, 2].

We also note the following simpler formula for the inverse of the ribbon element:

(4.8) $\qquad\qquad\qquad\qquad v^{-1} = \sum_{g \in G} e_g \# g.$

Recall the expression (3.10) for the canonical R-matrix $\mathcal{R} \in D^\omega G \otimes D^\omega G$.

LEMMA 4.4: *Let $n \geq 1$. Then $\mathcal{R}_n = \sum_{g \in G} e_g \otimes (g\beta^{-1})^n \beta^n$.*

Note, in addition, the following expression for the nth power g^n in $D^\omega G$ of an element $g \in G$:

$$g^n = \sum_{s \in G} \theta_s(g, g)\theta_s(g^2, g) \cdots \theta_s(g^{n-1}, g)e_s \# g^n.$$

Proof: Using relation (3.11), we compute

$$
\begin{aligned}
\mathcal{R}_n &= \mathcal{R}_{i_1}\mathcal{R}_{i_2} \cdots \mathcal{R}_{i_n} \otimes \mathcal{R}^{i_1}\mathcal{S}^2(\mathcal{R}^{i_2}) \cdots \mathcal{S}^{2n-2}(\mathcal{R}^{i_n}) \\
&= \mathcal{R}_{i_1}\mathcal{R}_{i_2} \cdots \mathcal{R}_{i_n} \otimes \mathcal{R}^{i_1}(\beta^{-1}\mathcal{R}^{i_2}\beta)(\beta^{-2}\mathcal{R}^{i_2}\beta^2) \cdots \beta^{-n+1}\mathcal{R}^{i_n}\beta^{n-1} \\
&= \mathcal{R}_{i_1}\mathcal{R}_{i_2} \cdots \mathcal{R}_{i_n} \otimes (\mathcal{R}^{i_1}\beta^{-1})(\mathcal{R}^{i_2}\beta^{-1}) \cdots (\mathcal{R}^{i_n}\beta^{-1})\beta^n,
\end{aligned}
$$

which, in view of (3.10), equals

$$\sum_{g_1, \ldots, g_n} e_{g_1} \ldots e_{g_n} \otimes (g_1\beta^{-1}) \cdots (g_n\beta^{-1})\beta^n = \sum_g e_g \otimes (g\beta^{-1})^n \beta^n,$$

as claimed. ∎

The exponent of $D^\omega G$ can be characterized in terms of the ribbon element (4.6).

LEMMA 4.5: *The exponent of $D^\omega G$ equals the smallest positive integer N such that $v^N \in G(D^\omega G)$.*

Proof: By Lemma 3.1 the exponent of $D^\omega G$ equals the order of $\mathcal{R}_{21}\mathcal{R}$. Since v is a ribbon element for $D^\omega G$, the lemma follows in view of formula (4.4). ∎

5. Exponent of group theoretical fusion categories

Let G be a finite group, and let $F \subseteq G$ be a subgroup. Let also $\omega\colon G \times G \times G \to k^\times$ be a normalized 3-cocycle, and $\tau\colon F \times F \to k^\times$ a normalized 2-cochain, such that $\omega|_{F \times F \times F} = d\tau$.

Consider the category Vec_ω^G of finite dimensional G-graded vector spaces, with associativity constraint given by ω. That is, Vec_ω^G is the representation category of the quasi-Hopf algebra (k^G, ω). Since the twisted group algebra $k_\tau F$ is an algebra in Vec_ω^G, the category $\mathcal{C}(G, \omega, F, \tau)$ of $k_\tau F$-bimodules in Vec_ω^G is a tensor category. A **group theoretical** category is by definition a fusion category equivalent to $\mathcal{C}(G, \omega, F, \tau)$ for some G, F, ω, τ [28, Section 3].

A (quasi)-Hopf algebra H is called group theoretical if $\mathrm{Rep}\, H$ is. By the results in [25], a quasi-Hopf algebra H is group theoretical if and only if its

quantum double is gauge equivalent to a Dijkgraaf–Pasquier–Roche quasi-Hopf algebra $D^\omega G$. In particular, if $\mathrm{Rep}\, H \simeq \mathcal{C}(G, \omega, F, \tau)$, then $\dim H = |G|$.

Properties (2.1) and (2.2) of the exponent imply the following lemma.

LEMMA 5.1: *Let $\mathcal{C} \simeq \mathcal{C}(G, \omega, F, \tau)$ be a group-theoretical category. Then $\exp \mathcal{C} = \exp D^\omega G = \exp(k^G, \omega)$.*

In particular if H is a group theoretical quasi-Hopf algebra, then $\exp H = \exp D^\omega G = \exp(k^G, \omega)$, for appropriate choice of a finite group G and a normalized 3-cocycle ω on G, such that $|G| = \dim H$.

Remark 5.2: Note that, by gauge invariance, the exponent of $D^\omega G$ depends only on the cohomology class of ω.

It is well-known that $|G| H^r(G, k^\times) = 0$, for all $r \geq 0$. However, this relation is not always true if we replace $|G|$ by $\exp G$. Nevertheless, we have the following weaker annihilation property. This will be used next to prove a divisibility property for the order of β in a twisted quantum double.

LEMMA 5.3: *Let $N = \exp G$. There exists a normalized 3-cocycle $\widetilde{\omega}$ which is cohomologous to ω and such that*

$$\widetilde{\omega}(g, g^{-1}, g)^N = 1, \quad \forall g \in G.$$

Proof: Since $|\langle g \rangle|$ divides the exponent of G, for all $g \in G$, then $\exp G$ anihilates $H^3(\langle g \rangle, k^\times)$, for all $g \in G$.

Therefore, for all $g \in G$, there exists a normalized 2-cochain $f^g \colon \langle g \rangle \times \langle g \rangle \to k^\times$ such that

$$\omega(x, y, z)^N = df^g(x, y, z) = f^g(xy, z) f^g(x, y) f^g(x, yz)^{-1} f^g(y, z)^{-1},$$

for all $x, y, z \in \langle g \rangle$. Because $\langle g^{-1} \rangle = \langle g \rangle$, we may choose f^g in a way such that $f^{g^{-1}} = f^g$.

Next we define a normalized 2-cochain $f \colon G \times G \to k^\times$ in the form

$$f(g, h) = \begin{cases} f^g(g, h), & \text{if } h \in \langle g \rangle, \\ 1, & \text{otherwise.} \end{cases}$$

Then, for all $g \in G$, we have

$$\begin{aligned} (df)(g, g^{-1}, g) &= f(e, g) f(g, g^{-1}) f(g, e)^{-1} f(g^{-1}, g)^{-1} = f(g, g^{-1}) f(g^{-1}, g)^{-1} \\ &= f^g(g, g^{-1}) f^{g^{-1}}(g^{-1}, g)^{-1} \\ &= df^g(g, g^{-1}, g) \quad (\text{since } f^g = f^{g^{-1}}) \\ &= \omega(g, g^{-1}, g)^N. \end{aligned}$$

The lemma is established by putting $\widetilde{\omega} = \omega d(f^{-1/N})$. ∎

As a consequence of Lemma 5.3 we obtain the following.

COROLLARY 5.4: *The order of β in the group of units of $D^\omega G$ divides the exponent of G.* ∎

Let G be a finite group and let ω be a normalized 3-cocycle on G. In what follows we shall give a proof of the characterization in Theorem 1.2.

Let $n \geq 1$. We introduce a map $\pi_{n,\omega} \colon G \to k^\times$, by the formula $\pi_1 = \epsilon$, and $\pi_{n,\omega}(g) = \pi_{n-1,\omega}(g)\omega(g, g^{n-1}, g)$, $g \in G$, $n \geq 2$. In other words,

$$(5.1) \qquad \pi_{n,\omega}(g) = \omega(g, g^{n-1}, g)\omega(g, g^{n-2}, g)\cdots\omega(g, g, g).$$

Compare with formula (A.3) in [6].

Let $g \in G$ and suppose that $|g|$ divides n. The following relation is easily seen and will be frequently used in what follows:

$$(5.2) \qquad \pi_{n,\omega}(g) = \pi_{|g|,\widetilde{\omega}}(g),$$

where $\widetilde{\omega} \colon \langle g \rangle \times \langle g \rangle \times \langle g \rangle \to k^\times$ is the 3-cocycle obtained from $\omega^{n/|g|}$ by restriction.

Recall the expression (4.8) for the inverse of the ribbon element in $D^\omega G$. The following lemma follows from a straightforward computation.

LEMMA 5.5: *Let $n \geq 1$. Then $v^{-n} = \sum_{g \in G} \pi_n(g) e_g \# g^n$.*

Let us denote by $\widehat{G}$ the group of (one-dimensional) characters on the group G. So that $\widehat{G} = G(k^G)$.

PROPOSITION 5.6:
  (i) *The exponent of G divides* $\exp D^\omega G$.
  (ii) *The exponent of $D^\omega G$ equals the smallest integer n with the properties*

$$\pi_{n,\omega} \in \widehat{G}, \quad g^n = e, \quad \forall g \in G.$$

Proof: By Lemma 4.5 the exponent of $D^\omega G$ equals the smallest positive integer n such that v^n (hence also v^{-n}) belongs to the group $G(D^\omega G)$.

Suppose that $v^n \in G(D^\omega G)$. By the description of group-like elements in twisted Drinfeld doubles from [22, Proposition 3.2] (cf. Subsection 3.1), there exist $x \in G$ and $f \in k^G$ such that $\gamma_x = df$ and

$$v^n = f \# x.$$

Applying the map $\epsilon \otimes \mathrm{id}$ to both sides, we get $(\epsilon \otimes \mathrm{id})(v^n) = \epsilon(f)x$. But $(\epsilon \otimes \mathrm{id})(v) = e$ because of formula (4.6), and $\epsilon \otimes \mathrm{id} \colon D^\omega G \to kG$ is an algebra

map. Therefore $\epsilon(f)x = e$. This implies that $x = e$, whence $df = \gamma_e = 1$, and thus $f \in \widehat{G}$. In particular, $v^n = f\#e \in k^G$ and $v^{-n} = f^{-1}\#e$. It follows from Lemma 5.5 that $g^n = e$, for all $g \in G$. Therefore $\exp G$ divides n. Hence part (i) follows.

Using that $g^n = e$, for all $g \in G$, Lemma 5.5 gives

$$v^{-n} = \sum_{g \in G} \pi_n(g)e_g \# g^n = \left(\sum_{g \in G} \pi_n(g)e_g \right) \# e.$$

Therefore π_n must be a character of G. This finishes the proof of the proposition. ∎

THEOREM 5.7: *The exponent of $D^\omega G$ divides $e(\omega)\exp G$.*

Proof: We may assume that $\omega(x, y, z)^{e(\omega)} = 1$, for all $x, y, z \in G$. It follows from equation (5.2) that

$$\pi_{e(\omega)\exp G, \omega} = 1.$$

This implies the theorem, in view of Proposition 5.6. ∎

Since both $e(\omega)$ and $\exp G$ divide $|G|$, it follows from Theorem 5.7 that for a twisted Drinfeld double $D^\omega G$ the exponent divides $|G|^2$ which equals the dimension of $D^\omega G$. We shall see next that this is not true in general for any quasi-Hopf algebra.

Example 5.8: Let $G = C_n = \langle a : a^n = 1 \rangle$ be a cyclic group of odd order n. The group $H^3(C_n, k^\times)$ is cyclic of order n parametrized by the cohomology classes of the 3-cocycles $\omega = \omega_\zeta$ defined by

(5.3) $$\omega(a^i, a^j, a^l) = \zeta^{lq_{ij}},$$

$0 \le i, j, l \le n - 1$, where $\zeta \in k^\times$ are the nth roots of 1, and $q_{ij} \in \mathbb{Z}$ is the quotient of $i + j$ in the division by n. Explicitly,

$$q_{ij} = \begin{cases} 0, & \text{if } i + j \le n - 1, \\ 1, & \text{if } i + j \ge n, \end{cases}$$

and

$$\omega(a^i, a^j, a^l) = \begin{cases} 1, & \text{if } i + j \le n - 1, \\ \zeta^l, & \text{if } i + j \ge n. \end{cases}$$

for all $0 \le i, j, l \le n - 1$.

LEMMA 5.9: *Let ω be as in (5.3). Then $\exp D^\omega C_n = |\zeta| n$.*

In particular, the exponent of the quasi-Hopf algebra (k^G, ω) should not divide the order of G (=its dimension).

Proof: Let $N = \exp D^\omega C_n$. By Theorem 5.7, $N/e(\omega)\exp C_n = |\zeta| n$. On the other hand, straightforward computations show that

$$\pi_{n,\omega}(a) = \prod_{j=1}^{n-1} \omega(a, a^j, a) = \zeta,$$

$$\pi_{n,\omega}(a^{n-1}) = \prod_{i=1}^{n-1} \omega(a^{n-1}, a^{n-i}, a^{n-1}) = \zeta^{(n-1)^2} = \zeta.$$

Since n/N, we have

$$\pi_{N,\omega}(a) = \pi_{n,\omega^{N/n}}(a) = \zeta^{N/n},$$

and

$$\pi_{N,\omega}(a^{n-1}) = \pi_{n,\omega^{N/n}}(a^{n-1}) = \zeta^{N/n}.$$

By Proposition 5.6 we must have $\zeta^{N/n} = \zeta^{-N/n}$, implying that $|\zeta|$ divides $2N/n$. Since $|\zeta|$ is odd by assumption, we obtain $|\zeta| n/N$. Hence $N = |\zeta| n$ as claimed. ∎

In the case $n = 2$, however, the exponent of $(k\mathbb{Z}_2, \omega)$, where ω is the non-trivial cocycle given by $\omega(a, a, a) = -1$, does equal the order of $\mathbb{Z}_2$, because in this case $\zeta = \zeta^{-1}$.

We now introduce a modified exponent $\exp_\omega G$ of a finite group G endowed with a 3-cocycle ω that will be useful to describe the exponent of $D^\omega G$. Let us denote by ω_g the restriction of ω to the subgroup generated by $g \in G$. Let

(5.4) $\exp_\omega G = \text{l.c.m.}[e(\omega_g)|g| : g \in G].$

It is clear that $\exp G$ divides $\exp_\omega G$, and $\exp_\omega G$ divides $e(\omega)\exp G$. For instance, when G is cyclic of order n, we have $\exp_\omega G = e(\omega)n$.

PROPOSITION 5.10: *The exponent of $D^\omega G$ divides $\exp_\omega G$. Moreover, equality holds when $|G|$ is odd.*

Proof: Let $n = \exp_\omega G$. Then $|g|/n$, for all $g \in G$, and by (5.2),

$$\pi_{n,\omega}(g) = \pi_{|g|,\tilde\omega},$$

for all $g \in G$, where $\widetilde{\omega}$ is the restriction to $\langle g \rangle$ of the 3-cocycle $\omega^{n/|g|}$. Since, by definition of $\exp_\omega G$, $e(\omega_g)$ divides $n/|g|$, then $\pi_{n,\omega}(g) = 1$, for all $g \in G$. By Proposition 5.6, $\exp D^\omega G$ divides n, as claimed.

Suppose now that $|G|$ is odd, and let $N = \exp D^\omega G$. Let $g \in G$. As in the proof of Lemma 5.9 we find that the class of the restriction of $\omega^{N/|g|}$ to the subgroup generated by g is trivial.

Therefore $e(\omega_g)|g|/N$, for all $g \in G$, implying that $\exp_\omega G/N$. This shows that $\exp D^\omega G = \exp_\omega G$, when $|G|$ is odd, as claimed.   ∎

Proof of Theorem 1.3:   If C is a cyclic group, then $|H^3(C, k^\times)| = |C|$, cf. Example 5.8. Then $e(\omega_g)/|g|$, for all $g \in G$. Thus $\exp_\omega G = \text{l.c.m.}[e(\omega_g)|g| : g \in G]$ divides $(\exp G)^2$. This implies the theorem in view of Proposition 5.10.   ∎

Example 5.11:   Suppose that $|G| = p^n$ and $\exp G = p^k$, where p is a prime number, and $2k \leq n$. Then $\exp D^\omega G$ divides $|G|$.

In particular, if G is an extraspecial p-group of order $|G| > p^3$, then $\exp D^\omega G$ divides $|G|$.

Proof: The first claim is clear from Corollary 5. Suppose that G is extraspecial and $|G| > p^3$. Then $|G| = p^{1+2m}$, for some integer $m \geq 2$, and by definition, there is a central extension $0 \to \mathbb{Z}_p \to G \to (\mathbb{Z}_p)^{2m} \to 1$; cf. [3, 8.23]. Note that, in general, if N is a normal subgroup of G, then $\exp G$ divides $\exp N \exp(G/N)$. Since both the kernel and the quotient are of exponent p, this implies that $\exp G$ divides p^2. Hence $(\exp G)^2/p^4$ and this divides $|G|$ because $|G| > p^3$.   ∎

Proof of Theorem 1.4:   To prove the theorem we shall combine the results in this section with Lemma 5.1 that tells us that $\exp \mathcal{C} = \exp D^\omega G$.

Part (i) follows from Proposition 5.6, (ii) follows from Theorem 5.7, (iii) follows from Theorem 1.3, and finally (iv) is a consequence of (i) and (iii).   ∎

Let us denote by $[G : g]$ the index in G of the subgroup generated by $g \in G$.

THEOREM 5.12: *Suppose that*
   (i) $e(\omega_g)$ *divides* $[G : g]$, *for all* $g \in G$. *Then*
   (ii) $\exp D^\omega G$ *divides* $|G|$.
   If the order of G is odd, then (i) is equivalent to (ii).

Proof: The first claim follows immediately from Proposition 5.10.

Now suppose that $|G|$ is odd and that Condition (ii) holds. As in the proof of Lemma 5.9 we find that the class of the restriction of $\omega^{[G:g]}$ to the subgroup generated by g is trivial. Then $e(\omega_g)$ divides $[G : g]$, and (i) holds.   ∎

5.1. GROUP THEORETICAL HOPF ALGEBRAS.   A group theoretical category is the representation category of a Hopf algebra if and only if it admits a fiber functor.

Recall from [28] that fiber functors of the group theoretical fusion category $\mathcal{C}(G, \omega, F, \alpha)$ are classified by equivalence classes of subgroups $\Gamma \subseteq G$ and 2-cocycles β on Γ such that

(1) $\omega|_\Gamma = 1$;

(2) $G = F\Gamma$;

(3) the cocycle $\alpha\beta^{-1}$ is non-degenerate on $F \cap \Gamma$.

In what follows we shall assume that $\mathcal{C} \simeq \mathcal{C}(G, \omega, F, \alpha)$ is a group-theoretical category admitting a fiber functor. That is, $\mathcal{C} \simeq \operatorname{Rep} A$, for some group-theoretical Hopf algebra A.

The exponent of a group theoretical Hopf algebra turns out to have a simpler description in terms of twisted Drinfeld double.

LEMMA 5.13: *Let $\pi_{n,\omega}$ be as in (5.1), $n \geq 1$. The following statements are equivalent:*

  (i) *$\pi_{n,\omega} \colon G \to k^\times$ is a group homomorphism;*

  (ii) *$\pi_{n,\omega}(g) = 1$, for all $g \in G$.*

Proof: We only need to show (i) $\Longrightarrow$ (ii). Let $\Gamma \subseteq G$ be a subgroup giving rise to a fiber functor. We may assume that $\omega|_\Gamma = 1$ and $\omega|_F = 1$.

Let $g \in G$. Then g can be written in the form $g = xs$, $x \in F$, $s \in \Gamma$. If $\pi_{n,\omega}$ is a group homomorphism, then $\pi_{n,\omega}(g) = \pi_{n,\omega}(x)\pi_{n,\omega}(s) = 1$.   ∎

PROPOSITION 5.14: *Let A be a group theoretical Hopf algebra with*

$$\operatorname{Rep} A \simeq \mathcal{C}(G, \omega, F, \alpha).$$

Then the exponent of A equals the order of the ribbon element v in $D^\omega G$.

Proof: By Lemma 5.1, $\exp A = \exp D^\omega G$. Combining Proposition 5.6 with Lemma 5.13, we see that $\exp D^\omega G$ equals the smallest integer n such that $g^n = e$, for all $g \in G$, and $\pi_{n,\omega} = 1$. This is exactly the order of v in view of formula (4.8).   ∎

Remark 5.15:   Since the index of a subgroup annihilates the kernel of the restriction map, we find that the following relation holds for every group theoretical Hopf algebra:

(5.5) $\exp D^\omega G / ([G : F]; [G : \Gamma]) \exp G.$

Next we prove that the results in Proposition 5.10 and Theorem 5.12 hold for group-theoretical Hopf algebras, without the assumption that G has odd order.

PROPOSITION 5.16: *We have* $\exp\mathcal{C} = \exp_\omega G$.

Proof: We know that $\exp\mathcal{C} = \exp D^\omega G$. The proof is identical to the proof of Proposition 5.10, using Lemma 5.13. ∎

Proof of Theorem 1.2: By Lemma 5.1, $\exp\mathcal{C} = \exp D^\omega G$. The theorem follows from Proposition 5.10 in case (i), and from Proposition 5.16 in case (ii). ∎

Remark 5.17: Note that, in general, it is not true that $\exp\mathcal{C}$ divides $\exp G$. For instance, if $\mathcal{C} = \operatorname{Rep} H_8$, where H_8 is the 8-dimensional Kac-Paljutkin Hopf algebra [17], then $G = D_4$ the dihedral group of order 8. We have in this case $\exp\mathcal{C} = 8$ while $\exp G = 4$.

We next give some necessary and sufficient conditions for $\exp\mathcal{C}$ to divide $\dim\mathcal{C}$.

THEOREM 5.18: *The following are equivalent:*
  (i) $\exp\mathcal{C}$ *divides* $\dim\mathcal{C}$;
  (ii) $e(\omega_g)$ *divides* $[G : g]$, *for all* $g \in G$.

Proof: Identical to the proof of Theorem 5.12, using Lemma 5.13 and the fact that $\exp\mathcal{C} = \exp D^\omega G$. ∎

LEMMA 5.19: *Suppose that* $\exp\mathcal{C}$ *divides* $\dim\mathcal{C}$. *Then* $e(\omega)$ *divides* $[G : g]^2$, *for all* $g \in G$.

Proof: By Lemma 5.13, $\pi_{|G|,\omega}(g) = 1$, for all $g \in G$. Using Equation (5.2), we have $1 = \pi_{|G|,\omega}(g) = \pi_{e(g),\widetilde{\omega}}(g)$, where $\widetilde{\omega}\colon \langle g\rangle \times \langle g\rangle \times \langle g\rangle \to k^\times$ is the 3-cocycle obtained from $\omega^{[G:g]}$ by restriction.

As in the proof of Lemma 5.9 we find that the class of the restriction of $\omega^{[G:g]}$ to the subgroup generated by g is trivial.

The composition

$$H^3(G, k^\times) \overset{\mathrm{res}}{\to} H^3(\langle g\rangle, k^\times) \overset{\mathrm{tr}}{\to} H^3(G, k^\times),$$

where tr denotes the transfer map, is multiplication by the index $[G : g]$ [5]. Therefore, we find that $\omega^{[G:g]^2} = 1$. Hence $e(\omega)/[G : g]^2$, as claimed. ∎

Suppose that $G = F\Gamma$ is any factorizable finite group. Let $\widetilde{H}^3(G, k^\times)$ denote the kernel of the restriction map $H^3(G, k^\times) \to H^3(F, k^\times) \oplus H^3(\Gamma, k^\times)$. The following question is of a purely cohomological nature.

QUESTION 5.20: *Does the product* $\exp \widetilde{H}^3(G, k^\times) \exp G$ *divide the order of* G?

An affirmative answer to this question would guarantee that the exponent conjecture holds true for all group-theoretical Hopf algebras.

5.2. ABELIAN EXTENSIONS.   The class of group-theoretical quasi-Hopf algebras contains in particular the class of abelian bicrossed product Hopf algebras, first studied by G. I Kac [16]. We refer the reader to [23, 24] for the main features of the subject.

In what follows we shall consider a fixed matched pair of finite groups (F, Γ) with respect to compatible actions $\rhd\colon \Gamma \times F \to F$, $\lhd\colon \Gamma \times F \to \Gamma$. These actions determine a unique group structure on the product of F with Γ, denoted $G = F \bowtie \Gamma$, in such a way that G admits an **exact factorization** $G = F\Gamma$, $F \cap \Gamma = 1$.

Remark 5.21:   For every group with an exact factorization as above, there are two convergent spectral sequences

$$H^p(F, H^q(\Gamma, k^\times)) \Rightarrow H^{p+q}(G, k^\times),$$
$$H^p(\Gamma, H^q(F, k^\times)) \Rightarrow H^{p+q}(G, k^\times).$$

These spectral sequences come from the double complex in [23, 24] whose total complex gives a free resolution of the G-module $\mathbb{Z}$.

For every class (σ, τ) in $\mathrm{Opext}(k^G, kF)$; that is, $\sigma\colon F \times F \to (k^\Gamma)^\times$ and $\tau\colon \Gamma \times \Gamma \to (k^F)^\times$ are normalized 2-cocycles subject to certain compatibility conditions, there is a bicrossed product Hopf algebra $A := k^{\Gamma \tau} \#_\sigma kF$. This gives a one-to-one correspondence between the equivalence classes of Hopf algebra extensions

(5.6) $k \to k^\Gamma \to A \to kF \to k,$

affording the actions $\rhd$, $\lhd$, and the abelian group $\mathrm{Opext}(k^\Gamma, kF)$.

The **Kac exact sequence** associated to the matched pair (F, Γ) [16, 23] has the following form:

$$0 \to H^1(G, k^\times) \xrightarrow{\mathrm{res}} H^1(F, k^\times) \oplus H^1(\Gamma, k^\times) \to \mathrm{Aut}(k^\Gamma \# kF)$$
$$\to H^2(G, k^\times) \xrightarrow{\mathrm{res}} H^2(F, k^\times) \oplus H^2(\Gamma, k^\times) \xrightarrow{\delta} \mathrm{Opext}(k^\Gamma, kF)$$
$$\to H^3(G, k^\times) \xrightarrow{\mathrm{res}} H^3(F, k^\times) \oplus H^3(\Gamma, k^\times) \to \ldots$$

where the res denote the restriction maps.

Hopf algebras A arising from abelian exact sequences are always group-theoretical: indeed, Rep $A \simeq \mathcal{C}(G, \omega, F, 1)$, where $G = F \bowtie \Gamma$ and ω is the image of (σ, τ) in Kac exact sequence; see [25, Theorem 1.3]. Explicitly, the 3-cocycle $\omega = \omega(\sigma, \tau)$ can be represented by

$$(5.7) \qquad \omega(xg, x'g', x''g'') = \tau_{x''}(g \triangleleft x', g')\sigma_g(x', g' \triangleright x''),$$

for all $x, x', x'' \in F$, $g, g', g'' \in \Gamma$.

Recall the Schur–Zassenhauss Theorem for finite groups that says that any extension G of a group F by a group Γ, with $|F|$ and $|\Gamma|$ relatively prime, splits. That is, G is a semidirect product of $F \rtimes \Gamma$. The following proposition gives an analogue of this result for Hopf algebras.

PROPOSITION 5.22: *Suppose that $|F|$ and $|\Gamma|$ are relatively prime. Let A be a Hopf algebra fitting into an extension* (5.6). *Then A is obtained from the split extension $k^\Gamma \# kF$ by twisting the multiplication and the comultiplication.*

Proof: Let (σ, τ) be the element in $\mathrm{Opext}(F, \Gamma)$ corresponding to the extension (5.6). The indexes $[G, \Gamma] = |F|$ and $[G, F] = |\Gamma|$ annihilate the kernel of the restriction map

$$H^3(G, k^\times) \to H^3(\Gamma, k^\times) \oplus H^3(F, k^\times),$$

whence by exactness of the Kac sequence, (σ, τ) belongs to the image of δ. Now the result of Masuoka on cocycle twists of bicrossed products [24] implies the proposition. ∎

COROLLARY 5.23: *Let the exact sequence* (5.6) *and suppose that $|F|$, $|\Gamma|$ are relatively prime. Then $\exp A = \exp F \bowtie \Gamma$.*

In particular, A and all their cocycle twists satisfy the exponent conjecture 1.1.

Proof: By Proposition 5.23 and the twist invariance of the exponent, we may assume that the extension (5.6) splits. In this case the 3-cocycle ω associated to A under the Kac exact sequence is trivial. The corollary follows from Theorem 1.3. ∎

The following is a consequence of Theorem 1.3.

COROLLARY 5.24: *Let A be a Hopf algebra which fits into an abelian exact sequence* (5.6). *Then $\exp A$ divides $\exp \mathrm{Opext}(\Gamma, F) \exp G$.*

Proof: By Theorem 1.3 (ii), $\exp A$ divides $e(\omega) \exp G$, where ω is the 3-cocycle coming from the element in $\mathrm{Opext}(F, \Gamma)$ corresponding to A under the map

δ in the Kac sequence. The corollary follows from the exactness of the sequence. ∎

The following proposition is a refinement of the relation (5.5) in the case of abelian exact sequences. It generalizes the statement in Corollary 5.22.

PROPOSITION 5.25: *Let A be a Hopf algebra which fits into an abelian exact sequence (5.6). Then $\exp A$ divides $(|F|; |\Gamma|)\exp G$.* ∎

Proof: The indexes $[G, \Gamma] = |F|$ and $[G, F] = |\Gamma|$ annihilate the kernel of the restriction map $H^3(G, k^\times) \to H^3(\Gamma, k^\times) \oplus H^3(F, k^\times)$. Hence $e(\omega)/(|F|; |\Gamma|)$. This implies the proposition. ∎

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